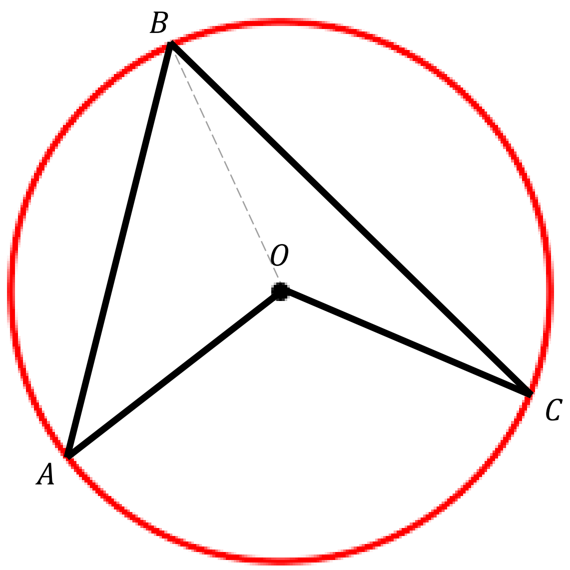
Circle Theorems

How to Quote - The angle at the centre is double the angle at the circumference or, the angle at the circumference is half the angle at the centre.

Hypothesis – Let $\angle ABC$ be an angle at the circumference and $\angle AOC$ the angle at the centre insisting on the same arc AC

Theses – $\angle AOC = 2 \times \angle ABC$



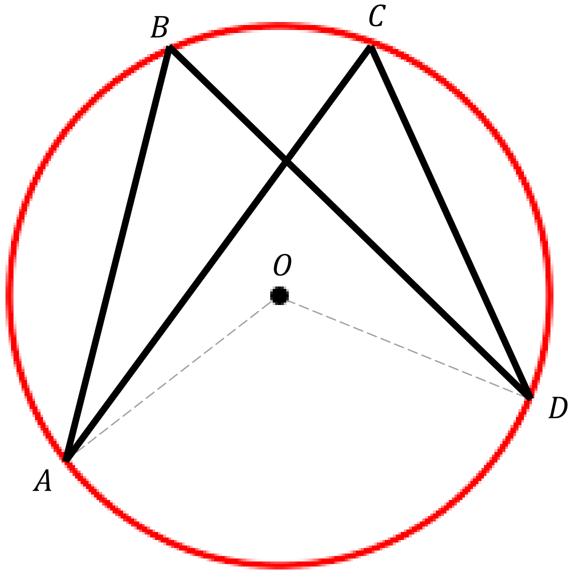
Proof – Let $\angle OBC = x$ and $\angle OBA = y$ so that $\angle ABC = x + y$
Consider triangle OBC:
$OC = OB$ because they are both radii. Therefore, triangle OBC is isosceles and consequently, $\angle OCB = \angle OBC = x$. Therefore, $\angle COB = 180° - 2x$, since angle in triangles add up to 180°.
Similarly, consider triangle ABO:
$OB = AO$ because they are both radii. Therefore, triangle AOB is isosceles and consequently, $\angle BAO = \angle ABO = y$. Therefore, $\angle AOB = 180° - 2y$, since angle in triangles add up to 180°.
Now, $\angle AOC = 360° - \angle COB - \angle AOB$, since angles at a point add up to 360°.
Substituting the expressions for $\angle AOB$ and $\angle COB$ obtained earlier in the equation above, we obtain:
$\angle AOC = 360° - (180° - 2x) - (180° - 2y) = 360° - 180° + 2x - 180° + 2y = 2x + 2y = 2(x + y) = 2 \times \angle ABC$

QED

How to Quote – Angles at the circumference insisting on the same arc are equal

Hypothesis – Let $\angle ABD$ and $\angle ACD$ be two angles at the circumference insisting on the same arc AD

Theses – $\angle ABD = \angle ACD$



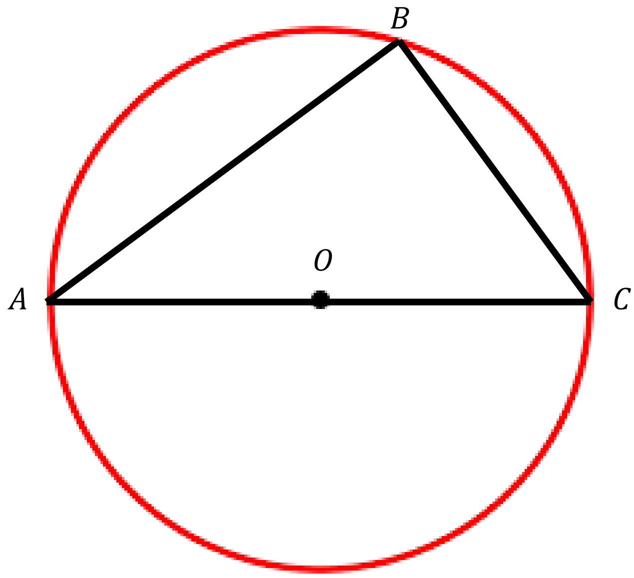
Proof – Let $\angle AOD$ be the angle the centre insisting on arc AD and let $\angle AOD = 2x$.
Then, $\angle ABD = x$, since the angle at the circumference is half the angle at the centre insisting on the same arc.
Similarly, $\angle ACD = x$, since the angle at the circumference is half the angle at the centre insisting on the same arc.
It follows that $\angle ABD = \angle ACD$

QED

How to Quote – Angles in a semicircle are right

Hypothesis – Let triangle ABC be inscribed in a semicircle with centre O and let AC be the diameter of the semicircle.

Theses – $\angle ABC = 90°$



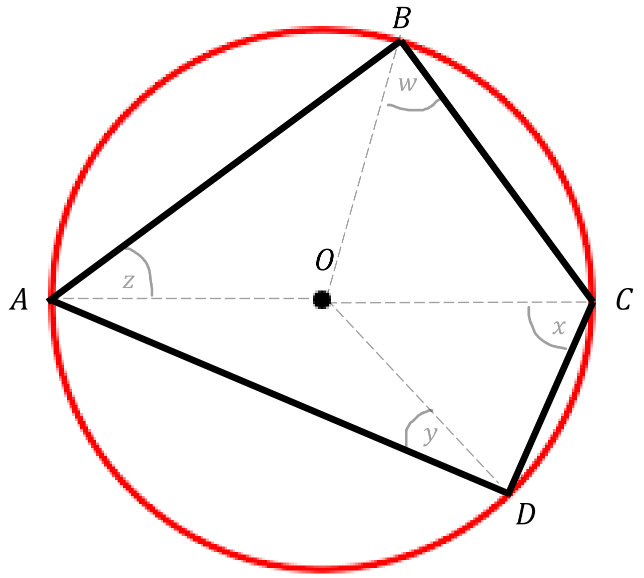
Proof – $\angle AOC$ is the angle at the centre insisting on arc AC.
Since the line AC is the diameter, $\angle AOC = 180°$.
Then, $\angle ABC = 90°$ since it's an angle at the circumference also insisting on arc AC and the angle at the circumference is half the angle at the centre insisting on the same arc.

QED

How to Quote – Opposite angles in cyclic quadrilaterals add up to 180°

Hypothesis – Let $ABCD$ be a quadrilateral inscribed in a circle of centre O (i.e. let $ABCD$ be a cyclic quadrilateral)

Theses – $\angle DAB + \angle BCD = 180°$ and $\angle ABC + \angle ADC = 180°$



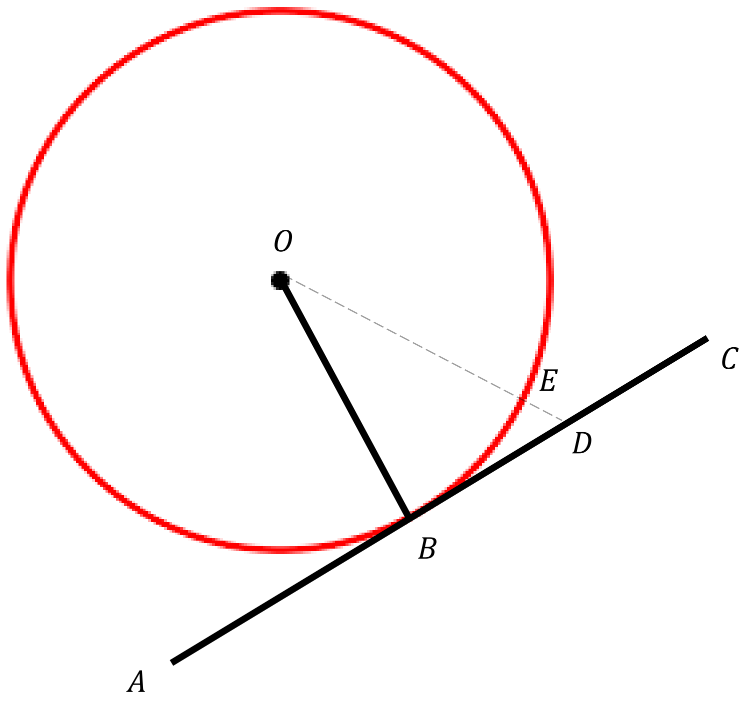
Proof – Let $\angle OBC = w$, $\angle OCD = x$, $\angle ODA = y$, $\angle OAB = z$.
Consider triangle OBC: $OB = OC$ since both radii. Therefore, triangle OBC is isosceles and consequently, $\angle BCO = \angle OBC = w$. In addition, $\angle BOC = 180° - 2w$, since angles in triangles add up to 180°.
Consider triangle OCD: $OC = OD$ since both radii. Therefore, triangle OCD is isosceles and consequently, $\angle ODC = \angle OCD = x$. In addition, $\angle DOC = 180° - 2x$, since angles in triangles add up to 180°.
Consider triangle ODA: $OD = OA$ since both radii. Therefore, triangle ODA is isosceles and consequently, $\angle OAD = \angle ODA = y$. In addition, $\angle AOD = 180° - 2y$, since angles in triangles add up to 180°.
Consider triangle OAB: $OA = OB$ since both radii. Therefore, triangle OAB is isosceles and consequently, $\angle ABO = \angle OAB = z$. In addition, $\angle AOB = 180° - 2z$, since angles in triangles add up to 180°.
Now, $\angle BCD = w + x$ and $\angle BAD = y + z$, by construction. So, $\angle BCD + \angle BAD = w + x + y + z$.
Now consider the angles at point O: $\angle BOC + \angle DOC + \angle AOD + \angle AOB = 360°$ since angles at a point add up to 360°. Substituting the expressions derived previously for the angles above, we obtain: $180° - 2w + 180° - 2x + 180° - 2y + 180° - 2z = 360°$.
Then, subtracting 720° from both sides and multiplying both sides by -1, we obtain
$2w + 2x + 2y + 2z = 360°$, finally dividing both sides by 2, we obtain
$w + x + y + z = \angle BCD + \angle BAD = 180°$. Since angles in quadrilaterals add up to 360°, it follows that $\angle ADC + \angle ABC = 180°$, also.

QED

How to Quote – The radius and the tangent are perpendicular

Hypothesis – Let point B be point lying on the circumference of a circle with centre O and let line ABC be the tangent to the circle at point B

Theses – $\angle OBC = 90°$



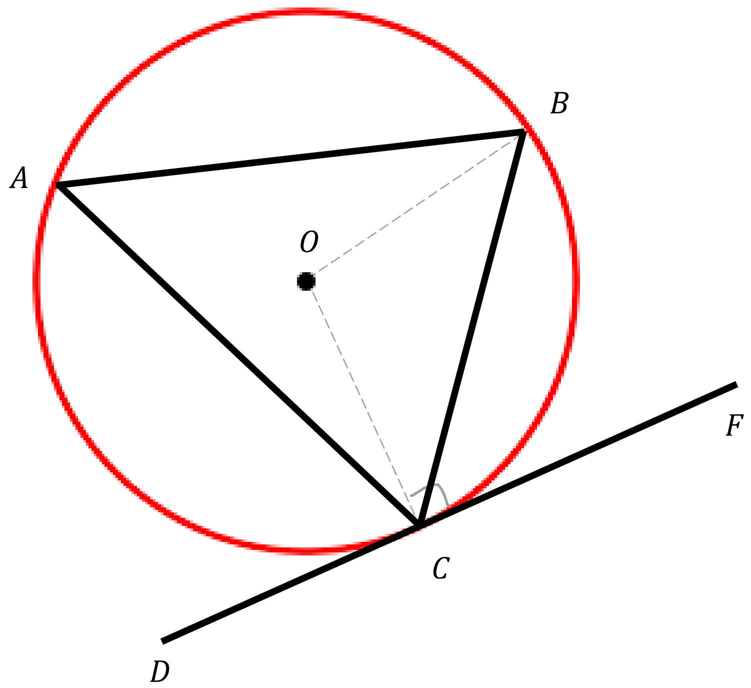
Proof (by contradiction) – Let's assume radius OB is not perpendicular to line AC. This implies that there must be another segment OD such that $\angle ODB = 90°$ and let E be the point at which OD intersects the circumference. Since we assumed that $\angle ODB = 90°$ then $\angle OBD$ and $\angle BOD$ both must be acute since angles in triangles add up to $180°$. Consequently, OB must be the longest side of triangle OBD since the longest side of a triangle always faces the largest angle. Therefore, we can write $OB > OD$. But $OB = OE$ since both radii, and therefore we must also have $OE > OD$. This is clearly a contradiction since $OD = OE + ED$, by construction.
We can therefore conclude that if OB is not not perpendicular to AC, then OB must be perpendicular to AC and $\angle OBC = 90°$

QED

How to Quote – Angles in alternate segments are equal

Hypothesis – Let ABC be a triangle inscribed in a circle with centre O and let line DCF be the tangent to the circle at point C

Theses – $\angle BCF = \angle BAC$



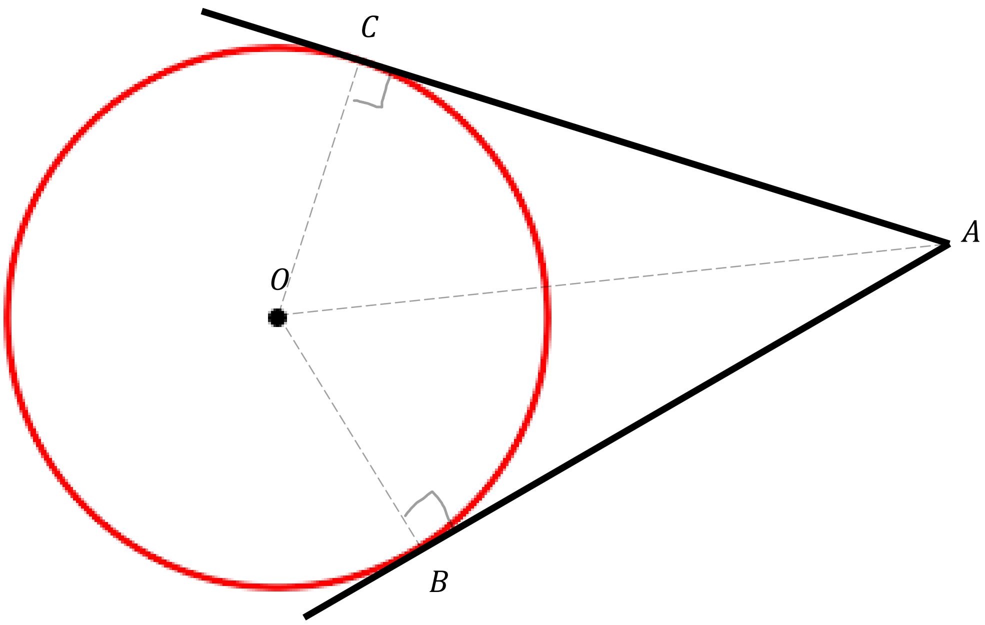
Proof – Let $\angle FCB = x$. Then $\angle OCB = 90° - x$ since the radius is perpendicular to the tangent. Consider triangle COB: $OC = OB$ since both radii. Therefore, triangle COB is isosceles and consequently $\angle OBC = \angle OCB = 90° - x$. Then, since angles in triangles add up to $180°$, $\angle COB = 180° - \angle OBC - \angle OCB = 180° - (90° - x) - (90° - x) = 180° - 90° + x - 90° + x = 2x$. $\angle COB = 2x$ is the angle at the centre insisting on arc CB. $\angle CAB$ is an angle at the circumference also insisting on arc CB. Therefore, $\angle CAB = x$, since the angle at the circumference is half the angle at the centre insisting on the same arc. It follows that $\angle BCF = \angle BAC$

QED

How to Quote – Tangents from the same point are equal in length

Hypothesis – Let point A be a point outside a circle with centre O, let B and C be points on the circumference and let AB and AC be the tangents to the circle at points B and C, respectively.

Theses – $AB = AC$



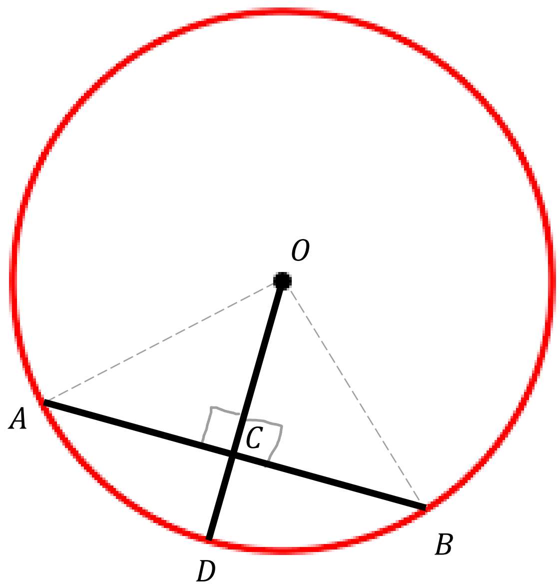
Proof – Consider triangles OBA and OCA: $OB = OC$, since they are both radii, $\angle OBA = \angle OCA = 90°$ because the radius is perpendicular to the tangent and hypotenuse AO is in common to both triangles. It follows that triangles OBC and OCA are congruent (since right angled triangles with congruent hypotenuse and one other side) and therefore $AB = AC$, since corresponding sides in congruent triangles.

QED

How to Quote – The perpendicular radius bisects the cord

Hypothesis – Let AB be a cord of a circle with centre O, let OD be the radius perpendicular to the cord AB and let C be point where OD intersects AB

Theses – $AC = CB$



Proof – Consider triangles AOC and BCO: $\angle BCO = \angle ACO = 90°$ by construction, hypotenuses OB and OA are equal since OB and OA are both radii, and OC is in common to both triangles. It follows that triangles ACO and BCO are congruent (since right angled triangles with congruent hypotenuse and one other side) and therefore, $AC = CB$ since corresponding sides in congruent triangles.

QED